

Klein's Paradox

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Abstract

We solve the one dimensional Feshbach-Villars equation for spin-1/2 particle subjected to a scalar smooth potential. The eight component wave function is given in terms of the hypergeometric functions and via a limiting procedure, the wave functions of the step potential are deduced. These wave functions are used to test the validity of the boundary conditions deduced from the Feshbach-Villars transformation. The creation of pairs is predicted from the boundary condition of the charge density.

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The eight component relativistic wave equation for spin-1/2 particle, called the Feshbach-Villars equation ($FV^{\frac{1}{2}}$), has been constructed and used to solve physical problems [1, 2, 3, 4]. The hydrogen atom is the first problem solved by Robson and Staudte [1, 2]; they found the same bound-state energy as the Dirac equation but the wave functions are different. Recently, Robson and Sutanto have solved the Compton scattering problem and found that the cross section is given, like in Dirac theory, by the Klein-Nishina formula [5]. The same authors have also calculated the transition probabilities for the Balmer and Lyman α -lines of hydrogenic atoms and compared them to the Dirac and Schrodinger results [6].

Following the idea of a previous paper [7], we study the one dimensional $FV^{\frac{1}{2}}$ equation for a particle subjected to a step potential $V(x)$. As we know the above problem is trivial in quantum mechanics but in relativistic quantum mechanics it emerges the famous Klein paradox. In addition to the known literature in this framework [8], we limit ourselves in this paper to illustrate the problem more clearly and justify the phenomenon of pair creation, and we left the problem of interpretation to the specialist researchers. The boundary conditions for the eight component wave function for the case of the step potential are unknown. Then, in order to bypass this problem, we take $V(x)$ as a smooth potential [9, 10, 11]. The analytic solution of the FV-1/2 equation with the smooth potential is given. We deduce, via a limiting procedure, the wave functions of the step potential and derive the transmission and reflection coefficients. The comparison with the Dirac coefficients constitutes the test to our calculations. The appropriate boundary conditions for the step potential are extracted from the Feshbach-Villars transformation. The boundary condition for the charge density is also evaluated. The validity of these boundary conditions is tested using the wave functions of the step potential. At the end, we discuss the boundary condition of the charge density and compare its predictions to those obtained from the transmission and reflection coefficients.

The Dirac equation can be written in a second-order form as

$$[(\gamma^\mu D_\mu)^2 + m^2]\Phi = 0, \quad (1)$$

where γ^μ are the Dirac matrices, D_μ is the minimally coupled derivative and $\Phi(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ is the four component wave function. This equation can be also written in the Klein-Gordon

form as follows [12]

$$[(D_\mu D^\mu + m^2)\mathbf{1}_4 + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}]\Phi = 0, \quad (2)$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The last term in Eq. (2) represents the spin interaction with the external electromagnetic field $F_{\mu\nu}$. By analogy with the equation of spin-0 particle, the second order equation (2) is called the Klein-Gordon equation for spin-1/2 particle ($KG_{\frac{1}{2}}$) [1, 2]. For spin-0 particles, this equation is reduced to a Klein-Gordon type equation.

In order to linearize Eq. (2) to a first order equation in time, the Feshbach-Villars linearization procedure is used to transform the four component wave function Φ to an eight component wave function ψ [1, 2]. The eight component wave function ψ satisfies a Schrodinger type equation [2]

$$H\psi = i\frac{\partial}{\partial t}\mathbf{1}_8\psi, \quad (3)$$

with

$$H = (\tau_3 + i\tau_2) \otimes \frac{1}{2m} \left[-\mathbf{D}^2\mathbf{1}_4 + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} \right] + m(\tau_3 \otimes \mathbf{1}_4) + eA_0\mathbf{1}_8, \quad (4)$$

where τ_2, τ_3 are the Pauli matrices, \otimes is the Kronecker(direct) product, $\mathbf{D} = \partial + ie\mathbf{A}$ is tridimensional minimally coupling and (\mathbf{A}, A_0) is the electromagnetic potential.

The Hamiltonian H is pseudo-Hermitian $H = \tau_4 H^\dagger \tau_4$ and the inner product is

$$(\psi, \psi) = \int \psi^\dagger \tau_4 \psi d^3V,$$

where $\tau_4 = \tau_3 \otimes \gamma^0$. This inner product is indefinite in sign, it takes positive or negative values. Then, the (FV-1/2) wave function space is not a Hilbert space and its dimension is twice that of the Dirac solution space [2].

In the Weyl representation of the gamma matrices, Eq. (3) separates into two four-component equations with the Hamiltonians

$$H_\xi = (\tau_3 + i\tau_2) \otimes \frac{1}{2m} \left[-\mathbf{D}^2\mathbf{1}_2 + ie\sigma \cdot (\mathbf{E} + i\mathbf{B}) \right] + m(\tau_3 \otimes \mathbf{1}_2) + eA_0\mathbf{1}_4, \quad (5)$$

$$H_\eta = (\tau_3 + i\tau_2) \otimes \frac{1}{2m} \left[-\mathbf{D}^2\mathbf{1}_2 - ie\sigma \cdot (\mathbf{E} - i\mathbf{B}) \right] + m(\tau_3 \otimes \mathbf{1}_2) + eA_0\mathbf{1}_4, \quad (6)$$

where \mathbf{E}, \mathbf{B} are the intensities of the electromagnetic field and $\sigma(\tau_1, \tau_2, \tau_3)$ are the Pauli matrices.

The Hamiltonians H_ξ and H_η satisfy also a Schrodinger type equation

$$H_\xi \psi_\xi = i\frac{\partial}{\partial t}\mathbf{1}_4\psi_\xi, \quad (7)$$

$$H_\eta \psi_\eta = i \frac{\partial}{\partial t} \mathbf{1}_4 \psi_\eta, \quad (8)$$

where ψ_ξ and ψ_η are four-component wave functions defined by their components as

$$\psi_\xi = (\psi_1, \psi_2, \psi_3, \psi_4)^T, \quad \psi_\eta = (\psi_5, \psi_6, \psi_7, \psi_8)^T,$$

and the eight component wave function is $\psi = (\psi_\xi, \psi_\eta)^T$. The Hamiltonians H_ξ and H_η , the wave functions ψ_ξ and ψ_η transform to each other under spatial inversion, respectively [1, 2].

In the Weyl representation of the gamma matrices, the density ρ is defined [2] as follows

$$\rho = \bar{\psi} \psi \quad \text{where} \quad \bar{\psi} = \psi^\dagger \tau_5, \quad \tau_5 = \tau_1 \otimes (\tau_3 \otimes \mathbf{1}_2),$$

where ρ and τ_5 are given by Eqs. (62),(63) in Ref. [2].

We define the one dimensional current j by

$$j = \frac{1}{2im} \left[\bar{\psi} O \frac{\partial \psi}{\partial x} - \frac{\partial \bar{\psi}}{\partial x} O \psi \right] - \frac{e}{m} A \bar{\psi} O \psi, \quad \text{where} \quad O = \mathbf{1}_2 \otimes (\tau_3 + i\tau_2) \otimes \mathbf{1}_2.$$

The values of ρ and j are independent of the representation and they satisfy the continuity equation. ρ is interpreted as the charge density of the particle. The positive solution ψ , the negative solution ψ_c , the charge density and the current are transformed by the charge conjugation as follows

$$\psi \longrightarrow \psi_c = \tau_1 \otimes \gamma^0 \psi^\dagger, \quad \rho \longrightarrow \rho_c = -\rho \quad \text{and} \quad j \longrightarrow j_c = j.$$

In one dimension, Hamiltonians (5),(6) for spin-1/2 particle subjected to a scalar potential $V(x)$ are

$$H_\xi = (\tau_3 + i\tau_2) \otimes \frac{1}{2m} \left[-\frac{d^2}{dx^2} \mathbf{1}_2 - ie\tau_1 \frac{dV(x)}{dx} \right] + m(\tau_3 \otimes \mathbf{1}_2) + eV(x) \mathbf{1}_4, \quad (9)$$

$$H_\eta = (\tau_3 + i\tau_2) \otimes \frac{1}{2m} \left[-\frac{d^2}{dx^2} \mathbf{1}_2 + ie\tau_1 \frac{dV(x)}{dx} \right] + m(\tau_3 \otimes \mathbf{1}_2) + eV(x) \mathbf{1}_4. \quad (10)$$

The terms $\pm ie\tau_1 \frac{dV(x)}{dx}$ represent the interaction of the spin with the derivative of the scalar potential.

The stationary solution has the form $\psi(x, t) = e^{-iEt} \psi(x)$ for each component of the wave function and Eq. (7) of the Hamiltonian H_ξ is equivalent to the following four coupled differential equations

$$\left[-\frac{d^2}{dx^2} + 2m^2 + 2meV(x) - 2mE \right] \psi_1 - \frac{d^2 \psi_3}{dx^2} - ie \frac{dV(x)}{dx} (\psi_2 + \psi_4) = 0, \quad (11)$$

$$\left[-\frac{d^2}{dx^2} + 2m^2 + 2meV(x) - 2mE \right] \psi_2 - \frac{d^2\psi_4}{dx^2} - ie\frac{dV(x)}{dx}(\psi_1 + \psi_3) = 0, \quad (12)$$

$$-\frac{d^2\psi_1}{dx^2} + \left[-\frac{d^2}{dx^2} + 2m^2 - 2meV(x) + 2mE \right] \psi_3 - ie\frac{dV(x)}{dx}(\psi_2 + \psi_4) = 0, \quad (13)$$

$$-\frac{d^2\psi_2}{dx^2} + \left[-\frac{d^2}{dx^2} + 2m^2 - 2meV(x) + 2mE \right] \psi_4 - ie\frac{dV(x)}{dx}(\psi_1 + \psi_3) = 0. \quad (14)$$

The difference of Eqs. (11)-(13) and (12)-(14) give, respectively,

$$\psi_1 - \psi_3 = \left[\frac{E - eV(x)}{m} \right] (\psi_1 + \psi_3), \quad (15)$$

$$\psi_2 - \psi_4 = \left[\frac{E - eV(x)}{m} \right] (\psi_2 + \psi_4). \quad (16)$$

Using these equations, the sum of Eqs. (11)-(13) and (12)-(14) give

$$\left[\frac{d^2}{dx^2} + [E - eV(x)]^2 - m^2 \right] (\psi_1 + \psi_3) + ie\frac{dV(x)}{dx}(\psi_2 + \psi_4) = 0, \quad (17)$$

$$\left[\frac{d^2}{dx^2} + [E - eV(x)]^2 - m^2 \right] (\psi_2 + \psi_4) + ie\frac{dV(x)}{dx}(\psi_1 + \psi_3) = 0. \quad (18)$$

The sum and the difference of the two last equations (17),(18) give

$$\left[\frac{d^2}{dx^2} + [E - eV(x)]^2 - m^2 + ie\frac{dV(x)}{dx} \right] \psi_\xi^s(x) = 0, \quad (19)$$

$$\left[\frac{d^2}{dx^2} + [E - eV(x)]^2 - m^2 - ie\frac{dV(x)}{dx} \right] \psi_\xi^d(x) = 0, \quad (20)$$

where

$$\psi_\xi^s(x) = \psi_1 + \psi_2 + \psi_3 + \psi_4, \quad \psi_\xi^d(x) = \psi_1 + \psi_3 - \psi_2 - \psi_4. \quad (21)$$

We note that the differential equation (19) is the same as the Dirac equation gives (Eq.(207.7) in Ref. [13]). If we take $V(x)$ as a step potential $V(x) = V_0\theta(x)$, we have in Eqs. (19),(20) the potential $V(x)$ and its derivative: $V'(x) = \frac{dV(x)}{dx} = V_0\delta(x)$. In the configuration space, the delta Dirac potential has particular treatment of its boundary conditions [14, 15]. In this paper, our goal is to find the wave functions without using boundary conditions. At the end, we use the obtained wave functions to test the validity of the boundary conditions deduced from the Feshbach-Villars transformation. For this, we use the scalar smooth potential $V(x)$ defined as

$$V(x) = \frac{V_0}{2}(1 + \tanh \frac{x}{2r}), \quad (22)$$

where V_0 and r are positive constants. In the limiting case $r \rightarrow 0$, $V(x) \rightarrow V_0\theta(x)$. It increases from the value $V = 0$ for $x = -\infty$ to the value $V = V_0$ for $x = +\infty$, the main rise occurring in the interval $-2r < x < +2r$: $V(-2r) = 0.1192V_0$, $V_0(2r) = 0.8807V_0$.

In order to find the solution of the differential equations (19),(20), we make the change of variable

$$y = \frac{1}{2}(1 - \tanh \frac{x}{2r}), \quad (23)$$

which maps the interval $x \in]-\infty, +\infty[$ to $y \in]0, 1[$. The new form of Eqs. (19), (20) are

$$\begin{aligned} \frac{1}{r^2}y^2(1-y)^2\frac{d^2\psi_\xi^s(y)}{dy^2} + \frac{1}{r^2}y(1-y)(1-2y)\frac{d\psi_\xi^s(y)}{dy} \\ + \left[(E + eV_0y - eV_0)^2 - m^2 + i\frac{eV_0}{r}y(1-y) \right] \psi_\xi^s(y) = 0, \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{1}{r^2}y^2(1-y)^2\frac{d^2\psi_\xi^d(y)}{dy^2} + \frac{1}{r^2}y(1-y)(1-2y)\frac{d\psi_\xi^d(y)}{dy} \\ + \left[(E + eV_0y - eV_0)^2 - m^2 - i\frac{eV_0}{r}y(1-y) \right] \psi_\xi^d(y) = 0. \end{aligned} \quad (25)$$

The singularities of these differential equations are $y = 0, 1, \infty$. Let us introduce the change $\psi_\xi^s(y) = y^\nu(1-y)^\mu f(y)$, and $\psi_\xi^d(y) = y^\nu(1-y)^\mu g(y)$, the last equations are reduced to the hypergeometric equation form

$$y(1-y)\frac{d^2f(y)}{dy^2} + [(2\nu+1) - y(2\nu+2\mu+2)]\frac{df(y)}{dy} - \left[(\mu+\nu+\frac{1}{2})^2 - \frac{v_1^2}{4} \right] f(y) = 0. \quad (26)$$

$$y(1-y)\frac{d^2g(y)}{dy^2} + [(2\nu+1) - y(2\nu+2\mu+2)]\frac{dg(y)}{dy} - \left[(\mu+\nu+\frac{1}{2})^2 - \frac{v_2^2}{4} \right] g(y) = 0, \quad (27)$$

where $\nu^2 = r^2[m^2 - (E - eV_0)^2]$, $\mu^2 = r^2(m^2 - E^2)$, $v_1^2 = 1 - 4r^2e^2V_0^2 + 4ireV_0$ and $v_2^2 = 1 - 4r^2e^2V_0^2 - 4ireV_0$.

In comparison with the spin-0 case, the imaginary part in the expressions of v_1^2 and v_2^2 represent the effect of the spin. If we remove them, Eqs. (26), (27) are reduced to the same equation as for the spin-0 case [7].

The general solutions of Eqs. (26),(27) are given in terms of the hypergeometric function

$$\begin{aligned} \psi_\xi^s(y) = & C_1 y^\nu(1-y)^\mu {}_2F_1(\mu+\nu+\frac{1}{2}-\frac{v_1}{2}, \mu+\nu+\frac{1}{2}+\frac{v_1}{2}, 1+2\nu, y) \\ & + C_{12} y^{-\nu}(1-y)^\mu {}_2F_1(\mu-\nu+\frac{1}{2}+\frac{v_1}{2}, \mu-\nu+\frac{1}{2}-\frac{v_1}{2}, 1-2\nu, y), \end{aligned}$$

$$\begin{aligned}\psi_\xi^d(y) &= D_1 y^\nu (1-y)^\mu {}_2F_1\left(\mu + \nu + \frac{1}{2} + \frac{v_2}{2}, \mu + \nu + \frac{1}{2} - \frac{v_2}{2}, 1 + 2\nu, y\right) \\ &\quad + D_{12} y^{-\nu} (1-y)^\mu {}_2F_1\left(\mu - \nu + \frac{1}{2} - \frac{v_2}{2}, \mu - \nu + \frac{1}{2} + \frac{v_2}{2}, 1 - 2\nu, y\right),\end{aligned}$$

where C_1 , C_{12} , D_1 and D_{12} are constants. We note that these solutions can be obtained directly from Eqs. (24), (25) using a symbolic software [16]. We have chosen the parameters of the hypergeometric function in order to have an analogy with the spin-0 solution [7]. An equivalent solution with others parameters is given by Eq. (207.15) in Ref. [13].

We choose the regular solutions at the origin $y = 0$

$$\psi_\xi^s(y) = C_1 y^\nu (1-y)^\mu {}_2F_1\left(\mu + \nu + \frac{1}{2} - \frac{v_1}{2}, \mu + \nu + \frac{1}{2} + \frac{v_1}{2}, 1 + 2\nu, y\right), \quad (28)$$

$$\psi_\xi^d(y) = D_1 y^\nu (1-y)^\mu {}_2F_1\left(\mu + \nu + \frac{1}{2} + \frac{v_2}{2}, \mu + \nu + \frac{1}{2} - \frac{v_2}{2}, 1 + 2\nu, y\right). \quad (29)$$

Then, the expressions of the components of the wave function ψ_ξ can be deduced as follows: from the definition (21) of $\psi_\xi^s(y)$ and $\psi_\xi^d(y)$, we have

$$\psi_\xi^s(y) + \psi_\xi^d(y) = 2(\psi_1 + \psi_3), \quad \psi_\xi^s(y) - \psi_\xi^d(y) = 2(\psi_2 + \psi_4),$$

and using the Eqs. (15),(16) we have

$$\begin{aligned}\psi_{1,3}(y) &= \frac{1}{4} \left[1 \pm \frac{E - eV(y)}{m} \right] [\psi_\xi^s(y) + \psi_\xi^d(y)], \\ \psi_{2,4}(y) &= \frac{1}{4} \left[1 \pm \frac{E - eV(y)}{m} \right] [\psi_\xi^s(y) - \psi_\xi^d(y)],\end{aligned} \quad (30)$$

where $\psi_\xi^s(y)$ and $\psi_\xi^d(y)$ are given by Eqs. (28),(29), the sign (+) corresponds to first index and the sign (−) to the second index.

The wave function ψ_η is calculated by the same method as ψ_ξ . We note that ψ_η can be also deduced from ψ_ξ under spatial inversion. It satisfies Eq. (8) where the hamiltonian H_η is given by Eq. (10) in the one dimensional case. Its components satisfy the four coupled differential equations (11)-(14) where the term $(-ie\frac{dV(x)}{dx})$ is replaced by the term $(+ie\frac{dV(x)}{dx})$ and the components $(\psi_1, \psi_2, \psi_3, \psi_4)$ by $(\psi_5, \psi_6, \psi_7, \psi_8)$, respectively. We give here the essential results:

$$\psi_5 - \psi_7 = \left[\frac{E - eV(x)}{m} \right] (\psi_5 + \psi_7), \quad (31)$$

$$\psi_6 - \psi_8 = \left[\frac{E - eV(x)}{m} \right] (\psi_6 + \psi_8), \quad (32)$$

$$\left[\frac{d^2}{dx^2} + [E - eV(x)]^2 - m^2 - ie \frac{dV(x)}{dx} \right] \psi_\eta^s(x) = 0, \quad (33)$$

$$\left[\frac{d^2}{dx^2} + [E - eV(x)]^2 - m^2 + ie \frac{dV(x)}{dx} \right] \psi_\eta^d(x) = 0, \quad (34)$$

where

$$\psi_\eta^s(x) = \psi_5 + \psi_6 + \psi_7 + \psi_8, \quad \psi_\eta^d(x) = \psi_5 + \psi_7 - \psi_6 - \psi_8. \quad (35)$$

We note that ψ_η^s and ψ_η^d satisfy the same differential equations as ψ_ξ^d and ψ_ξ^s , respectively. Then, the solutions of the differential equations (33) and (34) are similar to the solutions of Eqs. (20) and (19), respectively

$$\psi_\eta^s(y) = C_2 y^\nu (1-y)^\mu {}_2F_1\left(\mu + \nu + \frac{1}{2} + \frac{v_2}{2}, \mu + \nu + \frac{1}{2} - \frac{v_2}{2}, 1 + 2\nu, y\right), \quad (36)$$

$$\psi_\eta^d(y) = D_2 y^\nu (1-y)^\mu {}_2F_1\left(\mu + \nu + \frac{1}{2} - \frac{v_1}{2}, \mu + \nu + \frac{1}{2} + \frac{v_1}{2}, 1 + 2\nu, y\right), \quad (37)$$

where C_2 and D_2 are constants.

Then, the expressions of the components of ψ_η can be deduced as follows: from the definition (35) of $\psi_\eta^s(y)$ and $\psi_\eta^d(y)$, we have

$$\psi_\eta^s(y) + \psi_\eta^d(y) = 2(\psi_5 + \psi_7), \quad \psi_\eta^s(y) - \psi_\eta^d(y) = 2(\psi_6 + \psi_8),$$

and using Eqs. (31),(32) we have

$$\begin{aligned} \psi_{5,7}(y) &= \frac{1}{4} \left[1 \pm \frac{E - eV(y)}{m} \right] [\psi_\eta^s(y) + \psi_\eta^d(y)], \\ \psi_{6,8}(y) &= \frac{1}{4} \left[1 \pm \frac{E - eV(y)}{m} \right] [\psi_\eta^s(y) - \psi_\eta^d(y)], \end{aligned} \quad (38)$$

where $\psi_\eta^s(y)$ and $\psi_\eta^d(y)$ are given by Eqs. (36),(37).

We study now the asymptotic behavior of the wave function when $x \rightarrow \pm\infty$. First, when $x \rightarrow -\infty$ or $y \rightarrow 1$, we have $(1-y) \approx \exp(x/r)$; we use the property of the hypergeometric function which links the y and $(1-y)$ argument,

$${}_2F_1(a, b, c, y) = A {}_2F_1(a, b, a+b-c+1, 1-y) + B(1-y)^{c-a-b} {}_2F_1(c-a, c-b, c-a-b+1, 1-y),$$

with

$$A = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad B = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}.$$

The corresponding constants of the waves functions ψ_ξ^s and ψ_ξ^d are :

$$A_s = \frac{\Gamma(2\nu+1)\Gamma(-2\mu)}{\Gamma(\nu-\mu+\frac{1}{2}+\frac{v_1}{2})\Gamma(\nu-\mu+\frac{1}{2}-\frac{v_1}{2})}, \quad B_s = \frac{\Gamma(2\nu+1)\Gamma(2\mu)}{\Gamma(\mu+\nu+\frac{1}{2}-\frac{v_1}{2})\Gamma(\mu+\nu+\frac{1}{2}+\frac{v_1}{2})}, \quad (39)$$

$$A_d = \frac{\Gamma(2\nu+1)\Gamma(-2\mu)}{\Gamma(\nu-\mu+\frac{1}{2}-\frac{v_2}{2})\Gamma(\nu-\mu+\frac{1}{2}+\frac{v_2}{2})}, \quad B_d = \frac{\Gamma(2\nu+1)\Gamma(2\mu)}{\Gamma(\mu+\nu+\frac{1}{2}+\frac{v_2}{2})\Gamma(\mu+\nu+\frac{1}{2}-\frac{v_2}{2})},$$

$\lim_{y \rightarrow 1} y^\nu = 1$, $\lim_{y \rightarrow 1} (1-y)^\mu = e^{\mu x/r}$, $\lim_{y \rightarrow 1} (1-y)^{-\mu} = e^{-\mu x/r}$, and ${}_2F_1(a, b, c, 0) = 1$.

Thus, when $x \rightarrow -\infty$ or $y \rightarrow 1$, the waves functions ψ_ξ^s and ψ_ξ^d have the following behavior:

$$\psi_\xi^s(x) \xrightarrow{x \rightarrow -\infty} A_s e^{\mu x/r} + B_s e^{-\mu x/r}, \quad \psi_\xi^d(x) \xrightarrow{x \rightarrow -\infty} A_d e^{\mu x/r} + B_d e^{-\mu x/r}. \quad (40)$$

Setting $\mu = -ir k_1$, with $k_1^2 = E^2 - m^2$, where k_1 is real positive

$$\psi_\xi^s(x) \xrightarrow{x \rightarrow -\infty} A_s e^{-ik_1 x} + B_s e^{ik_1 x}, \quad \psi_\xi^d(x) \xrightarrow{x \rightarrow -\infty} A_d e^{-ik_1 x} + B_d e^{ik_1 x}. \quad (41)$$

For the limit when $x \rightarrow +\infty$, or $y \rightarrow 0$, ${}_2F_1(a, b, c, 0) = 1$, $\lim_{y \rightarrow 0} y^\nu = e^{-\nu x/r}$, and $\lim_{y \rightarrow 0} (1-y)^\mu = 1$.

Then, the waves functions have the following behavior:

$$\psi_\xi^s(x) \xrightarrow{x \rightarrow +\infty} e^{-\nu x/r}, \quad \psi_\xi^d(x) \xrightarrow{x \rightarrow +\infty} e^{-\nu x/r}. \quad (42)$$

Setting $\nu = -ir k_2$, with $k_2^2 = [(E - eV_0)^2 - m^2]$ while k_2 is real for $E < eV_0 - m$ or $E > eV_0 + m$ and k_2 is imaginary for $eV_0 - m < E < eV_0 + m$. Then, the waves functions are

$$\psi_\xi^s(x) \xrightarrow{x \rightarrow +\infty} e^{ik_2 x}, \quad \psi_\xi^d(x) \xrightarrow{x \rightarrow +\infty} e^{ik_2 x}. \quad (43)$$

By the same method the asymptotic behavior of the waves functions ψ_η^s and ψ_η^d are

$$\psi_\eta^s(x) \xrightarrow{x \rightarrow -\infty} A_d e^{-ik_1 x} + B_d e^{ik_1 x}, \quad \psi_\eta^d(x) \xrightarrow{x \rightarrow -\infty} A_s e^{-ik_1 x} + B_s e^{ik_1 x}, \quad (44)$$

$$\psi_\eta^s(x) \xrightarrow{x \rightarrow +\infty} e^{ik_2 x}, \quad \psi_\eta^d(x) \xrightarrow{x \rightarrow +\infty} e^{ik_2 x}. \quad (45)$$

We note here that for the plane waves and the other wave function which are not square-integrable the renormalization condition takes the form [2] $(\psi_E, \psi_{E'}) = \pm \delta(E - E')$, while for the square-integrable wave function it takes the form $(\psi, \psi) = \pm 1$.

The reflection and transmission coefficients can be calculated from the current density in the one dimensional case

$$j = \frac{1}{2im} \left[\bar{\psi} O \frac{\partial \psi}{\partial x} - \frac{\partial \bar{\psi}}{\partial x} O \psi \right], \quad A = 0, \quad (46)$$

where $O = \mathbf{1}_2 \otimes (\tau_3 + i\tau_2) \otimes \mathbf{1}_2$. Using the last definition of the current and the incident wave, we find that the incident current is

$$j_{inc} = \frac{k_1}{m} [B_s^* B_d + B_s B_d^*].$$

The reflected current is evaluated using the reflected wave

$$j_{ref} = -\frac{k_1}{m} [A_s^* A_d + A_s A_d^*].$$

Then, the reflection coefficient is

$$R = \frac{|j_{ref}|}{|j_{inc}|} = \frac{|A_s^* A_d + A_s A_d^*|}{|B_s^* B_d + B_s B_d^*|}. \quad (47)$$

The transmission coefficient is evaluated in terms of the transmitted wave,

$$j_{tr} = \frac{1}{m} (k_2 + k_2^\dagger) \exp i(k_2 - k_2^\dagger)x.$$

If k_2 is real,

$$j_{tr} = \frac{2k_2}{m},$$

and the transmission coefficient T is

$$T = \frac{|j_{tr}|}{|j_{inc}|} = \frac{2k_2}{k_1} \frac{1}{|B_s^* B_d + B_s B_d^*|}. \quad (48)$$

If k_2 is imaginary,

$$j_{tr} = 0, \quad T = 0 \quad \text{and} \quad R = 1, \quad (49)$$

in this case we have a total reflection.

We consider now the limiting case when the smooth potential tends to step potential, i.e., when the parameter r tends to 0.

For the first region $x < 0$, the limits of the coefficients (39) when $r \rightarrow 0^+$ are

$$\lim_{r \rightarrow 0^+} A_s = \frac{k_1 - k_2 - eV_0}{2k_1}, \quad \lim_{r \rightarrow 0^+} B_s = \frac{k_1 + k_2 + eV_0}{2k_1}, \quad (50)$$

$$\lim_{r \rightarrow 0^+} A_d = \frac{k_1 - k_2 + eV_0}{2k_1}, \quad \lim_{r \rightarrow 0^+} B_d = \frac{k_1 + k_2 - eV_0}{2k_1}, \quad (51)$$

and the waves functions $\psi_\xi^s(x)$ and $\psi_\xi^d(x)$ are

$$\psi_\xi^s(x) = C_1\theta(-x) \left\{ \left[\frac{k_1 - k_2 - eV_0}{2k_1} \right] \exp(-ik_1x) + \left[\frac{k_1 + k_2 + eV_0}{2k_1} \right] \exp(+ik_1x) \right\}, \quad (52)$$

$$\psi_\xi^d(x) = D_1\theta(-x) \left\{ \left[\frac{k_1 - k_2 + eV_0}{2k_1} \right] \exp(-ik_1x) + \left[\frac{k_1 + k_2 - eV_0}{2k_1} \right] \exp(+ik_1x) \right\}. \quad (53)$$

The terms $\pm eV_0$ in the above relations are the contribution of the spin and if we remove them, Eqs. (52),(53) are reduced to the same equation as for the spin-0 case [7].

For the second region $x > 0$, the waves functions are similar to Eqs. (43)

$$\psi_\xi^s(x) = C_1\theta(x)e^{ik_2x}, \quad \psi_\xi^d(x) = D_1\theta(x)e^{ik_2x}. \quad (54)$$

Then, the waves functions can be written in compact form for the two regions

$$\psi_\xi^s(x) = C_1\theta(-x) \left\{ \left[\frac{k_1 - k_2}{2k_1} - \frac{eV_0}{2k_1} \right] \exp(-ik_1x) + \left[\frac{k_1 + k_2}{2k_1} + \frac{eV_0}{2k_1} \right] \exp(+ik_1x) \right\} + C_1\theta(x)e^{ik_2x}, \quad (55)$$

$$\psi_\xi^d(x) = D_1\theta(-x) \left\{ \left[\frac{k_1 - k_2}{2k_1} + \frac{eV_0}{2k_1} \right] \exp(-ik_1x) + \left[\frac{k_1 + k_2}{2k_1} - \frac{eV_0}{2k_1} \right] \exp(+ik_1x) \right\} + D_1\theta(x)e^{ik_2x}. \quad (56)$$

Using the same method, the waves functions ψ_η^s and ψ_η^d in the two regions are

$$\psi_\eta^s(x) = C_2\theta(-x) \left\{ \left[\frac{k_1 - k_2}{2k_1} + \frac{eV_0}{2k_1} \right] \exp(-ik_1x) + \left[\frac{k_1 + k_2}{2k_1} - \frac{eV_0}{2k_1} \right] \exp(+ik_1x) \right\} + C_2\theta(x)e^{ik_2x}, \quad (57)$$

$$\psi_\eta^d(x) = D_2\theta(-x) \left\{ \left[\frac{k_1 - k_2}{2k_1} - \frac{eV_0}{2k_1} \right] \exp(-ik_1x) + \left[\frac{k_1 + k_2}{2k_1} + \frac{eV_0}{2k_1} \right] \exp(+ik_1x) \right\} + D_2\theta(x)e^{ik_2x}. \quad (58)$$

In the case of the step potential, we note that the presence of the delta Dirac in the differential equations (19),(20),(33),(34) implies that the functions (55)-(58) are continuous at $x = 0$ and their derivatives discontinuous. Then, the final expression of the wave function $\psi(x)$ of the step potential is deduced using the expressions of the eight components (30),(38) and the last relations (55)-(58):

$$\psi(x) = \begin{pmatrix} \psi_{1,3}(x) \\ \psi_{2,4}(x) \\ \psi_{5,7}(x) \\ \psi_{6,8}(x) \end{pmatrix} = \frac{1}{4} \left[1 \pm \frac{E - eV_0\theta(x)}{m} \right] \begin{pmatrix} \psi_\xi^s(y) + \psi_\xi^d(y) \\ \psi_\xi^s(y) - \psi_\xi^d(y) \\ \psi_\eta^s(y) + \psi_\eta^d(y) \\ \psi_\eta^s(y) - \psi_\eta^d(y) \end{pmatrix} \quad (59)$$

where the sign (+) corresponds to the first index and the sign (−) to the second index and

$$\begin{aligned}
\psi_\xi^s(y) \pm \psi_\xi^d(y) &= \theta(-x) \left\{ \left[(C_1 \pm D_1) \frac{k_1 - k_2}{2k_1} - \frac{eV_0}{2k_1} (C_1 \mp D_1) \right] \exp(-ik_1x) \right. \\
&\quad \left. + \left[(C_1 \pm D_1) \frac{k_1 + k_2}{2k_1} + \frac{eV_0}{2k_1} (C_1 \mp D_1) \right] \exp(+ik_1x) \right\} + \theta(x) (C_1 \pm D_1) e^{ik_2x}, \\
\psi_\eta^s(y) \pm \psi_\eta^d(y) &= \theta(-x) \left\{ \left[(C_2 \pm D_2) \frac{k_1 - k_2}{2k_1} + \frac{eV_0}{2k_1} (C_2 \mp D_2) \right] \exp(-ik_1x) \right. \\
&\quad \left. + \left[(C_2 \pm D_2) \frac{k_1 + k_2}{2k_1} - \frac{eV_0}{2k_1} (C_2 \mp D_2) \right] \exp(+ik_1x) \right\} + \theta(x) (C_2 \pm D_2) e^{ik_2x}.
\end{aligned}$$

We note also that the wave function of spin-0 particle [7, 17] can be deduced from the wave function (59) of spin-1/2 particle if we take only the two components (ψ_1, ψ_3) and remove the effect of the spin-1/2 term (eV_0). This analogy with spin-0 particle does not exist for the Dirac wave functions.

At the end, from Eqs. (47),(48) and (50),(51) we deduce the reflection coefficient R and the transmission coefficient T for the step potential

- For k_2 real positive ($k_2 > 0$) and $E > eV_0 + m$, we have

$$R = \frac{(k_1 - k_2)^2 - (eV_0)^2}{(k_1 + k_2)^2 - (eV_0)^2}, \quad T = \frac{4k_1k_2}{(k_1 + k_2)^2 - (eV_0)^2}, \quad \text{and} \quad R + T = 1. \quad (60)$$

- For k_2 real negative ($k_2 < 0$) and $m < E < eV_0 - m$, we have

$$R = \frac{(k_1 + k_2)^2 - (eV_0)^2}{(k_1 - k_2)^2 - (eV_0)^2}, \quad T = \frac{4k_1k_2}{(k_1 - k_2)^2 - (eV_0)^2}, \quad \text{and} \quad R - T = 1, \quad (61)$$

which is the well known Klein Paradox.

The coefficients T and R given in the above relations coincide exactly with the Dirac ones [13, 18, 19]. If we remove the effect of the spin-1/2 term (eV_0) from the above relations (60),(61) we find the reflection and the transmission coefficients of the spin-0 case [7].

Like in the spin-0 case [7], in the following we are going to look for boundary conditions using the Feshbach-Villars transformation. In the case of the step potential, the presence of the delta Dirac in Eq. (2) implies that the KG-1/2 wave function Φ is continuous at $x = 0$ and its derivative discontinuous.

The Feshbach-Villars transformation is defined for the eight components case as follows:

$$\psi_\xi(x, t) = \begin{pmatrix} \psi_1(x, t) \\ \psi_2(x, t) \\ \psi_3(x, t) \\ \psi_4(x, t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 + \frac{i}{m}(\frac{\partial}{\partial t} + ieV)\varphi_1 \\ \varphi_2 + \frac{i}{m}(\frac{\partial}{\partial t} + ieV)\varphi_2 \\ \varphi_1 - \frac{i}{m}(\frac{\partial}{\partial t} + ieV)\varphi_1 \\ \varphi_2 - \frac{i}{m}(\frac{\partial}{\partial t} + ieV)\varphi_2 \end{pmatrix}, \quad (62)$$

$$\psi_\eta(x, t) = \begin{pmatrix} \psi_5(x, t) \\ \psi_6(x, t) \\ \psi_7(x, t) \\ \psi_8(x, t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_3 + \frac{i}{m}(\frac{\partial}{\partial t} + ieV)\varphi_3 \\ \varphi_4 + \frac{i}{m}(\frac{\partial}{\partial t} + ieV)\varphi_4 \\ \varphi_3 - \frac{i}{m}(\frac{\partial}{\partial t} + ieV)\varphi_3 \\ \varphi_4 - \frac{i}{m}(\frac{\partial}{\partial t} + ieV)\varphi_4 \end{pmatrix}, \quad (63)$$

which can be written in the abridged two-component form

$$\begin{pmatrix} \psi_j \\ \psi_{j+2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_k + \frac{i}{m}(\frac{\partial}{\partial t} + ieV)\varphi_k \\ \varphi_k - \frac{i}{m}(\frac{\partial}{\partial t} + ieV)\varphi_k \end{pmatrix}, \quad (64)$$

where the couple (j, k) takes the following values: $(j, k) = \{(1, 1), (2, 2), (5, 3), (6, 4)\}$, respectively, $(\psi_j, \psi_{j+2})^T$ are the eight components of the wave function ψ with $j = 1, 2, 5, 6$ and $\varphi_k = \varphi_k(x, t)$ are the four components of the wave function Φ with $k = 1, 2, 3, 4$.

From the last definition (64) it follows that

$$\varphi_k = \frac{1}{\sqrt{2}}(\psi_j + \psi_{j+2}), \quad (65)$$

$$(i\frac{\partial}{\partial t} - eV)\varphi_k = \frac{m}{\sqrt{2}}(\psi_j - \psi_{j+2}). \quad (66)$$

The stationary KG-1/2 wave function Φ has the form $\Phi(x, t) = e^{-iEt}\Phi(x)$ for each component and the last equation is written as

$$(E - eV)\varphi_k = \frac{m}{\sqrt{2}}(\psi_j - \psi_{j+2}). \quad (67)$$

The continuity at $x = 0$ of the component φ_k of the KG-1/2 wave function defined in Eq. (65) gives

$$\psi_j(0^+) + \psi_{j+2}(0^+) = \psi_j(0^-) + \psi_{j+2}(0^-). \quad (68)$$

The continuity at $x = 0$ of the component φ_k of the KG-1/2 wave function defined in Eq. (67) gives

$$\psi_j(0^+) - \psi_{j+2}(0^+) = \frac{E - eV_0}{E} [\psi_j(0^-) - \psi_{j+2}(0^-)]. \quad (69)$$

From Eqs. (68) and (69) we can write the boundary conditions in the matrix form

$$\begin{pmatrix} \psi_j(0^+) \\ \psi_{j+2}(0^+) \end{pmatrix} = \begin{pmatrix} 1 - \frac{eV_0}{2E} & \frac{eV_0}{2E} \\ \frac{eV_0}{2E} & 1 - \frac{eV_0}{2E} \end{pmatrix} \begin{pmatrix} \psi_j(0^-) \\ \psi_{j+2}(0^-) \end{pmatrix}, \quad (70)$$

where $j = 1, 2, 5, 6$.

For the following particular cases the boundary conditions are more simple:

- For $E = \frac{eV_0}{2}$:

$$\psi_j(0^+) = \psi_{j+2}(0^-), \quad \psi_{j+2}(0^+) = \psi_j(0^-).$$

This case will be discussed at the end.

- For $E = eV_0$:

$$\psi_j(0^+) = \psi_{j+2}(0^+) = \frac{1}{2}[\psi_j(0^-) + \psi_{j+2}(0^-)].$$

The boundary conditions (70) are between the components of the wave function ψ . It can be written for all the eight components as follows: let us introduce the following notations

$$\psi_I = \sum_{j=1,2,5,6} \psi_j = \psi_1 + \psi_2 + \psi_5 + \psi_6,$$

$$\psi_{II} = \sum_{j=1,2,5,6} \psi_{j+2} = \psi_3 + \psi_4 + \psi_7 + \psi_8,$$

from relation (68), the sum of the eight components of the wave function satisfies

$$\psi_I(0^+) + \psi_{II}(0^+) = \psi_I(0^-) + \psi_{II}(0^-). \quad (71)$$

From relation (69), the difference between ψ_I and ψ_{II} satisfies

$$\psi_I(0^+) - \psi_{II}(0^+) = \frac{E - eV_0}{E} [\psi_I(0^-) - \psi_{II}(0^-)]. \quad (72)$$

From relations (71) and (72) we can also write the above boundary conditions in the matrix form

$$\begin{pmatrix} \psi_I(0^+) \\ \psi_{II}(0^+) \end{pmatrix} = \begin{pmatrix} 1 - \frac{eV_0}{2E} & \frac{eV_0}{2E} \\ \frac{eV_0}{2E} & 1 - \frac{eV_0}{2E} \end{pmatrix} \begin{pmatrix} \psi_I(0^-) \\ \psi_{II}(0^-) \end{pmatrix}. \quad (73)$$

Using the boundary conditions (70), we find that the charge density ρ is discontinuous

$$\rho(0^+) = \frac{E - eV_0}{E} \rho(0^-), \quad (74)$$

it gives the charge sign of the transmitted particle as a function of the energy E , the potential V_0 , the charge e and the charge density of the incident particle. We note that the same relation (74) can be also obtained for spin-0 particle. In the case of the Dirac equation, the density is continuous at $x = 0$. Then, the Dirac density is a probability density and cannot be a charge density .

At the end, we verify that the wave functions (59) of the step potential satisfy the boundary conditions (70),(73),(74) and the current(46) is continuous at $x = 0$.

The boundary conditions (71), (72) can be also interpreted by analogy with electromagnetic waves (when they traverse two different regions) as follows: the sum of the components of the wave function $\psi_s = \psi_I + \psi_{II}$ is continuous like the tangential component of the electric field but the difference of the two components $\psi_d = \psi_I - \psi_{II}$ is discontinuous like the normal component of the magnetic field.

Now we discuss the boundary condition (74) of the charge density and compare its predictions to those obtained from the transmission and reflection coefficients for the three following cases(we assume for all cases [18] that $eV_0 > 2m$ and $E > m$ i.e. $k_1 > 0$):

1. $E > eV_0 + m$

k_2 is real and $0 < \frac{E-eV_0}{E} < 1$, from Eq. (74) the charges densities $\rho(0^+)$ and $\rho(0^-)$ have the same sign, i.e. the charge of the transmitted particle has the same sign as the charge of the incident one. For $k_2 > 0$, the transmission and the reflection coefficients are given by relation (60). For $E \gg eV_0 + m$: $\rho(0^+) \approx \rho(0^-)$, $T \approx 1$ and $R \approx 0$. The two methods give the same results.

2. $eV_0 - m < E < eV_0 + m$

k_2 is imaginary and from Eq. (74) we consider two cases:

- For $eV_0 < E < eV_0 + m$: we have $\frac{E-eV_0}{E} > 0$; the charges densities $\rho(0^+)$ and $\rho(0^-)$ have the same sign.
- For $eV_0 - m < E < eV_0$: we have $\frac{E-eV_0}{E} < 0$; the charge densities $\rho(0^+)$ and $\rho(0^-)$ have opposite sign, this means that we have creation of particle-antiparticle pairs near the step barrier if the potential is strong enough.

On the other hand, from Eq. (49) we have a total reflection $R = 1$ and $T = 0$ and the wave function (59) is decreasing (evanescent wave) in the second region [19, 20]. The only case for which we have a total reflection from Eq. (74) is for $E = eV_0$: $\rho(0^+) = 0$.

3. $m < E < eV_0 - m$

k_2 is real and $\frac{E-eV_0}{E} < 0$, from Eq. (74) the charges densities $\rho(0^+)$ and $\rho(0^-)$ have opposite sign. This means that we have also creation of pairs near the step barrier. For $k_2 < 0$, the transmission and the reflection coefficients are given by Eqs. (61). Then, the creation of pairs in the Klein Paradox [19] is proved from the boundary condition of the charge density (74). We note that Guang-jiong Ni et al [20, 21] have discussed the Klein paradox, for the spin-0 case, using the current and the charge density in the two regions but they haven't used the boundary condition of the charge density.

Let us now study an interesting particular case for the spin-0 particle[7]:

- For the particular value $E = \frac{eV_0}{2}$ from the interval $m < E < eV_0 - m$ (we have assumed that $eV_0 > 2m$): we have $k_2 = \pm k_1$, $T = 1$ and $R = 0$ i.e. the incident particle is transmitted to the second region (called the resonance transmission) but from relation (74) we have $\rho(0^+) = -\rho(0^-)$: this means that we have creation of pairs near the step barrier and the transmitted particle is the antiparticle of the incident one. The result of the case $k_2 = -k_1$ need an interpretaion.

In summary, in order to find the wave functions of the step potential without the use of boundary conditions, we introduce the smooth potential as an intermediate stage. Then, we solve the one dimensional Feshbach-Villars equation for spin-1/2 particle subjected to the smooth potential. The eight-component wave function is given in terms of the hypergeometric functions. In the limiting case $r \longrightarrow 0$, the wave functions of the step potential are deduced in each region. The transmission and reflection coefficients are identical to the Dirac ones. We have also an analogy between the wave functions and the transmission and reflection coefficients of the spin-1/2 and the spin-0 particles. Boundary conditions relative to the step potential are extracted using the Feshbach-Villars transformation and the continuity of the KG-1/2 wave function at $x = 0$. The main result is that boundary conditions for the step potential are:

- the sum of the eight components $\psi_s = \psi_I + \psi_{II}$ is continuous:

$$\psi_s(0^+) = \psi_s(0^-),$$

- the difference of the two components $\psi_d = \psi_I - \psi_{II}$ is discontinuous:

$$\psi_d(0^+) = \frac{E - eV_0}{E} \psi_d(0^-),$$

- the charge density ρ is discontinuous:

$$\rho(0^+) = \frac{E - eV_0}{E} \rho(0^-),$$

and for $\frac{E - eV_0}{E} < 0$ we have creation of particle-antiparticle pairs if the potential is strong enough. Then, the number of particles becomes variable and this implies that we must use quantum field theory [8, 18, 22].

At the end, we note that we have omitted the singular solution in order to make comparison with the Dirac results. We propose to study the contributions of this solution separately to this paper.

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References

- [1] B.A. Robson, D.S. Staudte, J. Phys. A **29** (1996) 157.
- [2] D.S. Staudte, J. Phys. A **29** (1996) 167.
- [3] D.S. Staudte: " The physical interpretation of the eight-component relativistic spin-1/2 wave equation", unpublished.

- [4] D.S. Staudte: " A comparaisn of the eight-component spin-1/2 relativistic wave equation with other developments based upon the second order spin-1/2 equation ", unpublished.
- [5] B.A. Robson, S.H. Sutanto, Int. J. Theo. Phys. **40** (2001) 1491.
- [6] B.A. Robson, S.H. Sutanto: Int. J. Theo. Phys. **40** (2001) 1475.
- [7] M. Merad, L. Chetouani, A. Bounames, Phys. Lett. A **267** (2000) 225.
- [8] A. Calogeracos, N. Dombey, contemp. phys. **40** (1999) 313, arXiv: quant-ph/9905076.
- [9] L. Chetouani, L. Dekar, T.F. Hammann, Phys. Rev. A **52** (1995) 82.
- [10] M. Dekar, L. Chetouani, T.F. Hammann, J. Math. Phys. **39** (1998) 2551.
- [11] M. Dekar, L. Chetouani, T.F. Hammann, Phys. Rev. A **59** (1999) 107 .
- [12] R.P. Feynman, M. Gell-Mann, Phys. Rev. **109**(1958) 193.
- [13] S. Flugge, Pratical Quantum Mechanics, Vol. II, 2nd ed., Springer, Berlin, 1994, pp. 213-219.
- [14] C.L. Roy, Phys. Rev. A **47** (1993) 3417.
- [15] R.D. Benguria, H. castillo, M. Loewe, J. Phys. A **33** (2000) 5315, arXiv: quant-ph/0003045
- [16] Maple V Release 5, Waterloo Maple Inc., 1997.
- [17] T. Boudjedaa, L. Chetouani, M. Merad, Il Nuovo Cimento B **114** (1999) 1261.
- [18] N. Dombey, A. Calogeracos, Phys. Rep. **315** (1999) 41.
- [19] W. Greiner, Relativistic Quantum Mechanics, Springer, Berlin, 1990, pp. 261-267.
- [20] G-j. Ni, W-m Zhou, J. Yan, e-print arXiv: quant-ph/9905044.
- [21] G-j. Ni, H. Guan, W-m. Zhou, J. Yan, Chin. Phys. Lett. **17** (2000) 393, arXiv: quant-ph/0001016.
- [22] W. Greiner, Relativistic Quantum Mechanics, Springer, Berlin, 1990, p. 35.